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Electromagnetic waves with E parallel to B

John E Gray

Code G-71, Naval Surface Warfare Center, Dahlgren, VA 22448, USA

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Abstract. In this paper, two explanations of how to obtain electromagnetic waves with $\boldsymbol{E}\|\boldsymbol{B}$ are presented. The nature of these solutions is discussed, with a caveat on what constitutes a general solution. Finally, an example of an electromagnetic wave with the electric field parallel to magnetic field is presented.

1. Introduction

Students in electromagnetic theory have been taught throughout all levels of their education that the only plane wave solutions to Maxwell's equations are those with the electric field perpendicular to the magnetic field. This is a universally accepted 'myth' that is taught in the sophomore level texts [1], junior–senior level texts [2–5] and graduate level texts [6–8]. Recently, an article by Chu [9] has purportedly arrived at a general plane wave solution to Maxwell's equations with $\boldsymbol{E}\|\boldsymbol{B}$. Several authors [10, 11] have subsequently published refutations of this claim that miss the point. Rebuttals of the refutations have been published [13, 14] and several articles have appeared in the physics education literature [13, 15] attempting to explain this result. In this paper, an interpretation of electromagnetic waves with $\boldsymbol{E}\|\boldsymbol{B}$ is presented, with a caveat on what constitutes a general solution.

2. Discussion

A point that has not been discussed in the literature is the nature and interpretation of the method for obtaining Chu's solutions, rather the emphasis has been on how they can be obtained. We briefly show how the solutions were obtained in order to clarify discussion of the interpretation of the solution. If one assumes a harmonic time variation, with $k=\omega/c$, then instead of solving the Helmholtz equations for the electric and magnetic fields, the single equation

$$(\nabla^2+k^2)\boldsymbol{F}_k(\boldsymbol{r})=0 \tag{1}$$

can be used to obtain the electric and magnetic fields. The superposition vector

$$\boldsymbol{F}_k(\boldsymbol{r})=c_1\boldsymbol{A}_k(\boldsymbol{r})+c_2\nabla\times\boldsymbol{A}_k(\boldsymbol{r}) \tag{2}$$

is a linear combination of the electric and magnetic fields. Since c_1 and c_2 are arbitrary constants, any choice for them will also satisfy (1). The particular combination of coefficients $c_1=1$ and $c_2=1/k$ makes the superposition vector obey the differential relation

$$\nabla\times\boldsymbol{F}_k=k\boldsymbol{F}_k. \tag{3}$$

This is the relationship for the superposition vector that Chu assumes. By interchanging $\boldsymbol{F}$ for $\boldsymbol{A}$ in (2), the vector potential can be expressed in analogous fashion to be represented by

$$A_k(r) = \frac{1}{k}\nabla \times F_k(r) + F_k(r) \tag{4}$$

where $\boldsymbol{F}$ is an arbitrary vector obeying (3). The radiation gauge condition gives $\boldsymbol{k}\cdot\boldsymbol{A} = 0$, which also implies $\boldsymbol{k}$ is perpendicular to the superposition vector, so the fields are TEM. The fields are

$$E_k(r, t) = \frac{-\partial A_k(r, t)}{\partial t} = -j\omega\left(F_k(r, t) + \frac{1}{k}\nabla \times F_k(r)\right) = -j\omega A_k(r, t) \tag{5}$$

$$B_k(r, t) = \nabla \times A_k(r, t) = \nabla \times \left(F_k(r) + \frac{1}{k}\nabla \times F_k(r)\right) = kA_k(r, t). \tag{6}$$

From (5) and (6) it is clear that $\boldsymbol{E}(\boldsymbol{r}, t)\cdot\boldsymbol{B}(\boldsymbol{r}, t)$ is not equal to zero, the fields are not perpendicular to each other.

The vector differential equation

$$\nabla \times a = ka \tag{7}$$

that produces Chu's $\boldsymbol{E}\|\boldsymbol{B}$ solutions is a mathematically consistent equation which has a general solution [12]

$$a = \nabla \times (c\Psi) + \frac{1}{k}\nabla \times \nabla(c\Psi) \tag{8}$$

where $\boldsymbol{c}$ is a constant vector and Ψ is a solution to

$$\nabla^2\Psi + k^2\Psi = 0. \tag{9}$$

Regardless of the mathematical correctness of (7) or equivalently (3), the physical validity of the solutions is determined by what assumptions what into the derivation of (3) and how general the solution is.

An important point that is not always emphasized is what a general solution to a physical problem is. The term general means the equations used to obtain a solution and the solution itself are the widest possible solutions that can be used to fit arbitrary but physically realizable boundary conditions. The plane wave solutions to Maxwell's equations in free space, except for the harmonic time dependence, are general solutions. Free space means the absence of boundary conditions. The solutions obtained by Chu are not free space solutions, so are not in the same general class of solutions as the classical $\boldsymbol{E}\perp\boldsymbol{B}$ solutions. $\boldsymbol{F}$ was obtained by combinging the wave equations for the vector potential and magnetic field into one equation, equation (1), using linear superposition. Specific values for the superposition coefficients ($c_1 = 1$, $c_2 = 1/k$) were chosen so that $\boldsymbol{F}$ obeyed the differential relation in (3). The specific choice of the superposition coefficients is a boundary condition (specifically a Dirichlet boundary condition).

The classical $\boldsymbol{E}\perp\boldsymbol{B}$ solution to Maxwell's equations have no such boundary condtions used in their derivation, hence are general solutions. This is a point that has been lacking in the interpretation of the $\boldsymbol{E}$ parallel to $\boldsymbol{B}$ solutions. Standing wave solutions to Maxwell's equations are not surprising when boundary conditions are imposed. What is surprising is the claim by some authors that these are free space solutions.

This claim is what makes these solutions unexpected. Unless very general boundary conditions are considered, the class of solutions is pathological when compared to the class of admissible solutions. An example of a pathological solution is to compare the D'Alembertian solution to the wave equation $w(x, t) = f(x - ct)$ to the harmonic solution $w(x, t) = A\sin(kx - wt)$. The harmonic solution is a pathological solution, though an important solution, of the wave equation. Generalizations of the harmonic solution are obtained by making A a function of k, i.e. $A = A(k)$, and integrating over all possible k.

To clarify the point that the Chu type solutions are the result of boundary conditions, a specific example is shown. Plane wave solutions can be obtained for $\boldsymbol{F}$ by noting space is isotropic along the direction of propagation, a vector z is chosen along the direction of $\boldsymbol{k}$ so that $\boldsymbol{k} = kz$. For propagation in the z direction, the operator ∇^2 becomes d^2/dz^2, so the solution takes the form

$$\boldsymbol{F}(z) = \boldsymbol{C}_1 \cos(kz) + \boldsymbol{C}_2 \sin(kz) \tag{10}$$

where

$$\boldsymbol{C}_1 = \langle c_{11}, c_{12}, 0\rangle \qquad \boldsymbol{C}_2 = \langle c_{21}, c_{22}, 0\rangle. \tag{11}$$

Requiring (10) to obey (3) imposes the boundary conditions

$$c_{12} = c_{21} \qquad c_{11} = -c_{22} \tag{12}$$

so the solution for the superposition vector is

$$\boldsymbol{F}_1(z, t) = [\langle c_{11}, c_{12}, 0\rangle \cos(kz) + \langle c_{12}, -c_{11}, 0\rangle \sin(kz)]\, e^{j\omega t} \tag{13}$$

and the fields are

$$\boldsymbol{E} = -j\omega \boldsymbol{F}_1(z, t) \qquad \boldsymbol{B} = k\boldsymbol{F}_1(z, t). \tag{14}$$

These boundary conditions are identical to those for a wave along a string with fixed endpoints, a standing wave. The analogous type of boundary for electromagnetic radiation is to confine it between two perfectly reflecting boundaries (the electromagnetic equivalent of a string hanging between two infinitely heavy walls).

Electromagnetic waves with $\boldsymbol{E} \| \boldsymbol{B}$ are possible as long as they are confined to an enclosed region. This is similar to setting up a standing wave in a cavity resonator. One possible way to produce them is split a light beam in order to confine the beam between two mirrors, as discussed theoretically in [13, 14]. Actual experiments have been conducted by Bretenaker and Le Flock [16] using lasers to produce several different standing wave solutions with $\boldsymbol{E} \| \boldsymbol{B}$.

Zaghloul [13, 15] has claimed to have generalized Chu's conditions for $\boldsymbol{E} \| \boldsymbol{B}$. This claim at a generalization is false. The conditions required by Zaghloul,

$$\boldsymbol{A}_1'(\eta) \neq \pm \boldsymbol{A}_2'(\zeta) \tag{15}$$

and

$$|\boldsymbol{A}_1'(\eta)| = |\boldsymbol{A}_2'(\zeta)| \tag{16}$$

are boundary conditions, hence are not in the same class of solutions to Maxwell's equations as the classical $\boldsymbol{E} \perp \boldsymbol{B}$ solutions. This condition is more restrictive than the Chu solution. Requiring the magnitudes of two vector-valued functions to be equal along two opposite signed caustic surfaces is an even more restrictive boundary condition than Chu's result, so Zaghloul's claim to have generalized Chu's result is true only in a formal sense. The class of functions that will obey (15) and (16) is even more restrictive than those that will obey (15)-(16) is even more restrictive than those that obey (13).

4. Conclusion

This paper provides an explanation of a controversy in the well-explored field of Maxwell's equations. Contrary to what is implicitly assumed, electromagnetic waves with the electric field parallel to the magnetic field are not general solutions to Maxwell's equations in the same sense of the conventional $\boldsymbol{E} \perp \boldsymbol{B}$ solutions. The Chu result is the solution to a boundary value problem, hence is in no sense in the same class of solutions to Maxwell's equations as the classical $\boldsymbol{E} \perp \boldsymbol{B}$ solution are. The first solution is obtained by application of the principle of superposition that fixes the values of the arbitrary constants used in a general solution to the Helmholtz equation. The second solution is even more restrictive than the first, because it fixes functions along two different parametric lines of spacetime. These standing wave solutions to Maxwell's equations are not surprising, however, when boundary conditions are imposed, and with this understanding, the $\boldsymbol{E} \| \boldsymbol{B}$ solutions are not unexpected.

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